Anomalous mobility of Brownian particles in a tilted symmetric sawtooth potential

Astrid Haljas,[1,2] Romi Mankin,[1] Ako Sauga,[1,*] and Eerik Reiter[3]
[1]*Department of Natural Sciences, Tallinn Pedagogical University, 25 Narva Road, 10120 Tallinn, Estonia*
[2]*Institute of Theoretical Physics, Tartu University, 4 Tähe Street, 51010 Tartu, Estonia*
[3]*Institute of Physics, Tallinn Technical University, 5 Ehitajate Road, 19086 Tallinn, Estonia*
(Received 19 May 2004; published 28 October 2004)

Overdamped motion of Brownian particles in a 1D periodic system with a simple symmetric sawtooth potential subjected to both unbiased thermal noise and spatially nonhomogeneous three-level colored noise is considered analytically. Upon application of a tilting force the particles exhibit anomalous transport properties, namely, absolute negative mobility, negative differential mobility, and the phenomenon of hypersensitive differential response. It is established that the mobility (differential mobility included) depends nonmonotonically on the parameters (switching rate, amplitude, and temperature) of nonequilibrium and thermal noises. The necessary conditions for various anomalous transport properties are found.

DOI: 10.1103/PhysRevE.70.041107 PACS number(s): 05.40.−a, 05.60.−k, 02.50.−r

I. INTRODUCTION

The idea that noise, via its interaction with the nonlinearity of the system, can give counterintuitive results, has lead to many important discoveries: stochastic resonance [1], resonant activation [2], nonequilibrium phase transitions[3], and stochastic ratchets (Brownian motors) [4–11], to name but a few.

Recently, noise-induced anomalous transport phenomena of Brownian particles in nonlinear periodic structures have been the topic of a number of physical investigations. Among them, we can mention such as the ratchet effect, hypersensitive response, absolute negative mobility (ANM), and negative differential resistance (NDR).

The recent fashion for the ratchet effect, i.e., a directed motion of Brownian particles induced by nonequilibrium fluctuations, with no macroscopic driving applied, in spatially periodic structures has started with Magnasco's theoretical work [5]. The initial motivation in this field has come from cell biology, in particular from the studies of the mechanism of vesicle transport inside eukariotic cells [4–6]. Beyond that, it was suggested that the ratchet mechanism can be used for obtaining efficient separation methods of nanoscale objects, e.g., DNA molecules, proteins, viruses, etc. [4,7,8]. To date, the feasibility of particle transport by man-made devices has been experimentally demonstrated for several ratchet types [8–10]. Many different forms of ratchet systems are possible. The classification of different types of ratchets (correlation, flashing, etc.) is in Ref. [4]. Among them, we can mention the multidimensional ratchets in which by choosing different potentials, a variety of flow patterns from laminar drifts to rotation can be generated [11].

The motivation to study a hypersensitive response has come from numerical, analytical and experimental studies of a nonlinear Kramers oscillator with multiplicative white noise. Under the effect of intense multiplicative noise the system is able to amplify an ultrasmall deterministic ac signal [12]. Afterwards, a related phenomenon such as noise-induced hypersensitive transport was found in some other systems with multiplicative colored noise. It was shown that in such a system a macroscopic flux (current) of matter appears under the influence of an ultrasmall dc driving [13,14].

A characteristic feature of models with ANM is that upon the application of an external static force F, these models respond with a current that always runs in the direction opposite to that of the force (if the force is small enough) [15–23]. Notably, for $F=0$ no current appears due to spatial symmetry of the system. The effect of ANM is distinct from the phenomenon of negative differential mobility (or resistance) which is, for a sufficiently large F, characterized by a decrease of the current as the driving force F increases, but the system does not exhibit ANM [24]. Devices that display both ANM and negative differential resistance exist and they have important biophysical and technological applications (for a reference survey, see [15,21]), e.g. semiconductor devices [22,25], tunnel junction in superconductor devices [23], biological ion channels [24,26], etc. In these cases, the effect of ANM has a quantum-mechanical origin [22,23,27].

There are several categories of theoretical models for classical ANM. First, for interacting Brownian particles ANM was found as a genuine cooperative effect [16]. The authors of Ref. [17] suggested, on the basis of Ising-type variants of those models, that at least three cooperating units are sufficient for ANM. Second, in [15,18] various 2D spatial geometries have been proposed in which a single Brownian particle displays ANM. Recently, two interesting basic 1D models for a single Brownian particle exhibiting ANM were presented [19,20]. Although the mechanism of ANM in these 1D models is different, there is similarity in some aspects, notably, switching between different potential configurations (states) can take place only when the particle passes specific positions. For example, for the spatially continuous model in [19] the corresponding positions are the minima of the periodic potential with two minima per period. Note that as a result of the localized transitions between the different states, dependence of the stationary current on the switching rate disappears [19].

*Electronic address: ako@audentes.ee

As the "three-layer" basic diffusion mechanism for ANM described in [19] is robust enough and can be easily realized experimentally, we were to apply a similar approach to 1D models, where the transitions between different potential configurations are not localized at discrete points, i.e., the transitions appear rather in finite intervals. It is of interest, both from theoretical and practical viewpoints, to know whether such a modification of the basic model can cause novel unexpected effects.

In this article, we consider one-dimensional overdamped dynamical systems, where Brownian particles move in a spatially periodic piecewise linear symmetric potential, which has one minimum per period. The applied force consists of an additive static force and of a noise term composed of thermal noise with a temperature D and a nonequilibrium three-level colored noise with a switching rate ν. The three-level noise $Z(t)$, taking the values $z_n \in \{f, 0, -f\}$, is assumed to be spatially nonhomogeneous, so that transitions between the states $z_1=f$, $z_2=0$ and between the states $z_2=0$, $z_3=-f$ can take place only in the left half-period and in the right half-period of the potential, respectively.

Our purpose is to report some interesting phenomena, which take place in simple one-dimensional systems described above, arising as a consequence of interplay between a nonequilibrium noise, thermal noise, and deterministic force. On the basis of an exact expression for the current we have found a number of cooperation effects: (i) a resonant-like behavior of ANM at intermediate values of the switching rate; the presence and intensity of ANM can be controlled by the switching rate and by temperature, (ii) the existence of a negative differential resistance, (iii) for large values of the switching rate ν and a low temperature D the current is, at some values of the tilting force F, very sensitive to a small variation of F—a phenomenon called hypersensitive differential response (HDR), and in the region of HDR the value of differential mobility can be controlled by means of thermal noise; (iv) for certain system parameters, there is a finite interval of the tilting force where the current is very small as compared to that in the surroundings [the effect of "disjunct windows" (DW)]. It seems that the behavior mentioned last is a new anomalous transport phenomenon for Brownian particles.

We emphasize that to our knowledge such a rich variety of anomalous transport effects have never been reported before for an overdamped Brownian particle in a 1D periodic structure with a simple symmetric potential (with one minimum per period).

It is known [4–8] that the models with the ratchet effect are characterized by a nonzero current if the tilting force is absent ($F=0$), and, thus, they inevitably involve some kind of asymmetry. In contrast to that our present model with the effects of ANM, NDR, and HDR is symmetric for $F=0$.

The structure of the paper is as follows. In Sec. II we present the model investigated in this work. A master equation description of the model is given and a corresponding exact stationary solution is discussed. In Sec. III we analyze the behavior of the current. Omitting the rather lengthy analytical exact solution and referring instead to the representative results depicted in figures, we will analyze the behavior of the current at different limits, such as the slow-noise limit, fast-noise limit, low-temperature case, etc. Section IV contains brief concluding remarks. Some formulas are delegated to the Appendix.

II. A MODEL WITH A THREE-LEVEL NOISE

We consider overdamped motion of Brownian particles in a one-dimensional spatially periodic potential $\tilde{V}=\tilde{V}(\tilde{x}+L)$ of a period L and barrier height $\tilde{V}_0=\tilde{V}_{max}-\tilde{V}_{min}$. The additional force consists of thermal noise $\tilde{\xi}(\tilde{t})$ with a temperature T, a colored three-level Markovian noise $\tilde{Z}(\tilde{x},\tilde{t})$, and an external static force $\tilde{F}$. The system is described by the stochastic differential equation

$$\kappa\frac{d\tilde{X}}{d\tilde{t}} = -\frac{d\tilde{V}(\tilde{X})}{d\tilde{X}} + \tilde{F} + \tilde{\xi}(\tilde{t}) + \tilde{f}\tilde{Z}(\tilde{X},\tilde{t}), \qquad (1)$$

where κ is the friction coefficient and $\tilde{f}$ is a constant force. The thermal fluctuations $\tilde{\xi}(\tilde{t})$ are modeled by a zero-mean Gaussian white noise with the correlation function $\langle\tilde{\xi}(t_1)\tilde{\xi}(t_2)\rangle = 2\kappa k_B T\delta(\tilde{t}_1-\tilde{t}_2)$, where k_B is the Boltzmann constant. The term $\tilde{Z}(\tilde{X},\tilde{t})$ represents spatially nonhomogeneous fluctuations assumed to be a three-level Markovian stochastic process taking the values $\tilde{z}_1=-1$, $\tilde{z}_2=0$, $\tilde{z}_3=1$. The probabilities $W_n(t)$ that $\tilde{Z}(\tilde{X},\tilde{t})$ is in the state n at the time $\tilde{t}$ evolve according to the master equation

$$\frac{d}{d\tilde{t}}W_n(\tilde{t}) = \sum_{m=1}^{3}\tilde{U}_{nm}W_m(\tilde{t}), \qquad (2)$$

where

$$\tilde{\mathbf{U}} = \frac{\tilde{\nu}}{2}\begin{pmatrix} -\tilde{a}_1(\tilde{x}), & \tilde{a}_1(\tilde{x}), & 0 \\ \tilde{a}_1(\tilde{x}), & -1, & \tilde{a}_2(\tilde{x}) \\ 0, & \tilde{a}_2(\tilde{x}), & -\tilde{a}_2(\tilde{x}) \end{pmatrix} \qquad (3)$$

and $\tilde{a}_1(\tilde{x})=\Theta(\tilde{x}-L/2)$, $\tilde{a}_2(\tilde{x})=\Theta(L/2-\tilde{x})$; $\Theta(x)$ is the Heaviside function. By applying a scaling of the form

$$X=\frac{\tilde{X}}{L}, \quad V(x)=\frac{\tilde{V}(\tilde{x})}{\tilde{V}_0}, \quad t=\frac{\tilde{t}}{\tilde{t}_0}, \quad \xi=\frac{L\tilde{\xi}}{\tilde{V}_0}, \quad F=\frac{L}{\tilde{V}_0}\tilde{F},$$

$$f=\frac{L}{\tilde{V}_0}\tilde{f}, \qquad (4)$$

we get a dimensionless formulation of the dynamics with the potential V with the property $V(x)=V(x-1)$. By the choice of $t_0=\kappa L^2/\tilde{V}_0$ the dimensionless friction coefficient turns to unity. The rescaled noises are given by

$$\nu=\frac{\kappa L^2\tilde{\nu}}{\tilde{V}_0}, \quad a_1(x)=\Theta\left(x-\frac{1}{2}\right), \quad a_2(x)=\Theta\left(\frac{1}{2}-x\right),$$

$$D = \frac{k_B T}{\widetilde{V}_0}, \tag{5}$$

where $2D$ is the strength of the rescaled zero-mean Gaussian white noise $\xi(t)$. For brevity's sake, from now on we shall call D temperature. The dimensionless dynamics reads as

$$\frac{dX}{dt} = -\frac{dV(X)}{dX} + F + \xi(t) + fZ(X,t), \tag{6}$$

where the rescaled nonequilibrium noise $Z(X,t)$ is characterized by the transition matrix $\mathbf{U}$ of the form

$$\mathbf{U} = \frac{\nu}{2}\begin{pmatrix} -a_1(x), & a_1(x), & 0 \\ a_1(x), & -1, & a_2(x) \\ 0, & a_2(x), & -a_2(x) \end{pmatrix}.$$

Let us observe that the three-level stochastic process $Z(X,t)$ with property (5) is actually independent from the position variable x in the intervals $\left(0,\frac{1}{2}\right)$ mod 1 and $\left(\frac{1}{2},1\right)$ mod 1, and the transition probability per unit time $U_{nm}^{(i)}$, $i=0,1$, $n,m=1,2,3$, for a flipping from the state z_m into the state z_n factorizes, i.e.,

$$U_{nm}^{(i)} = Q_i(z_n)\nu_i(z_m), \quad n \neq m,$$

where indices $i=0$ and $i=1$ denote the intervals $\left(0,\frac{1}{2}\right)$mod 1 and $\left(\frac{1}{2},1\right)$mod 1, respectively; $Q_0(1)=Q_0(0)=\frac{1}{2}$, $Q_0(-1)=0$, $\nu_0(1)=\nu_0(0)=\nu$, $\nu_0(-1)=0$, $Q_1(0)=Q_1(-1)=\frac{1}{2}$, $Q_1(1)=0$, $\nu_1(0)=\nu_1(-1)=\nu$, $\nu_1(1)=0$. It means that the process $Z(X,t)$ can be presented as the sum of two kangaroo processes $Z(X,t)=Z_0(t)\delta_{0,1}+Z_1(t)\delta_{1,i}$ [28]. If the kangaroo processes $Z_0(t)$ and $Z_1(t)$ are statistically independent, which is the case addressed in the present paper, then the two-dimensional process $\{x(t),z(t)\}$ is Markovian and its joint probability density $P_n(x,t)$ for the position variable $x(t)$ and the fluctuation variable $z(t)$ obeys the master equation of the form (see Ref. [28])

$$\frac{\partial}{\partial t}P_n(x,t) = -\frac{\partial}{\partial x}\left[\left(h(x) + F + z_n f - D\frac{\partial}{\partial x}\right)P_n(x,t)\right] + \sum_m U_{nm}P_m(x,t), \tag{7}$$

with $m,n=1,2,3$; $z_1=-1$, $z_2=0$, $z_3=1$; $h(x)=-dV(x)/dx$. More precisely, in the interval $(0,1)$ the master equation (7) splits up into two differential equations $\mathbf{P}_i(x,t)=(P_{1i},P_{2i},P_{3i})(i=0,1)$, defined in the intervals $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$, respectively.

The stationary current J is then evaluated via the current densities

$$j_n(x) = \left(h(x) + z_n f + F - D\frac{\partial}{\partial x}\right)P_n^s(x), \quad J = \sum_n j_n(x), \tag{8}$$

where $P_n^s(x)$ is the stationary probability density for the state (x,z_n). It follows from Eq. (7) that the current J is constant. In the stationary case, the total net probability flux between the states $n=1,2,3$ must vanish, implying

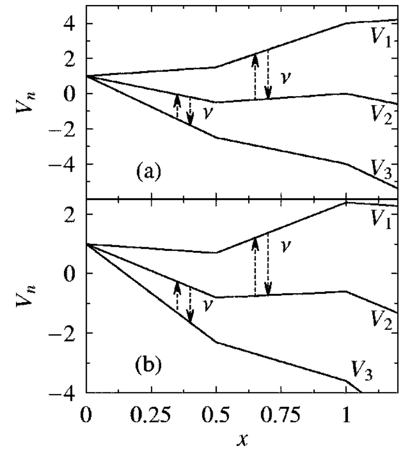

FIG. 1. Schematic representation of different states and their transitions in the model (6) with the sawtooth-like potential (10) at low temperatures. The lines depict the net potentials $V_n(x)=V(x)-Fx-z_n fx$ with $z_1=-1$, $z_2=0$, and $z_3=1$. Unbiased transitions with a switching rate ν can take place between the discrete states, but only at specific positions, namely, in the interval $x\in\left(0,\frac{1}{2}\right)$, modulo 1 between V_2 and V_3, and in the interval $x\in\left(\frac{1}{2},1\right)$ modulo 1 between V_2 and V_1. All quantities are dimensionless with scaling by Eqs. (4) and (5). (a) The case of $F=1$, $f=4$. (b) The case of $F=1.6$, $f=3$.

$$\int_0^{1/2} P_2^s(x)dx = \int_0^{1/2} P_3^s(x)dx,$$

$$\int_{1/2}^{1} P_1^s(x)dx = \int_{1/2}^{1} P_2^s(x)dx. \tag{9}$$

To derive an exact formula for J, we present an analysis of the system of Eq. (6) for a piecewise linear sawtooth-like symmetric potential,

$$V(x) = \begin{cases} -(2x-1), & x \in (0,1/2) \bmod 1, \\ (2x-1), & x \in (1/2,1) \bmod 1. \end{cases} \tag{10}$$

The force $h(x)=-dV(x)/dx$ being periodic, the stationary distributions $P_n^s(x)$ as solutions of Eqs. (7) are also periodic and it suffices to consider the problem in the interval $[0,1)$. The force corresponding to the potential of Eq. (10) is

$$h(x) = \begin{cases} h_0 := 2, & x \in (0,1/2), \\ h_1 := -2, & x \in (1/2,1). \end{cases} \tag{11}$$

A schematic representation on the three configurations assumed by the "net potentials" $V_n(x)=V(x)-Fx-z_n fx$ associated with the right-hand side of Eq. (6) is shown in Fig. 1. Regarding the symmetry of the dynamical system (6), we notice that $J(-F)=-J(F)$. Thus, we may confine ourselves to the case $F \geqslant 0$. Obviously, for $F=0$ the system is effectively isotropic and no current can occur.

As the force $h(x)$ is piecewisely constant Eq. (7) splits up into two linear differential equations with constant coefficients for the two vector functions $\mathbf{P}_i^s(x)=(P_{1i}^s,P_{2i}^s,P_{3i}^s)(i=0,1)$ defined on the intervals $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$, respectively. The solution reads as

$$P_{ni}^{s} = A_{ni0} + \sum_{k=1}^{4} C_{ik} A_{nik} \exp\left[\frac{\lambda_{ik}(F)x}{D}\right], \tag{12}$$

where A_{ni0} and A_{nik} are given by

$$A_{ni0} = C_{i0} + \delta_{n,2i+1}\frac{J - 3C_{i0}(h_i + F)}{\lambda_{i1}(F)}, \quad A_{ni1} = \delta_{n,2i+1},$$

$$A_{2i+1\ i\ j} = 0, \quad A_{2ij} = 1,$$

$$A_{3-2i\ i\ j} = 1 + \frac{2\lambda_{ij}(F)}{D\nu}[h_i + F - \lambda_{ij}(F)], \quad j = 2,3,4,$$

C_{i0}, C_{ik} are constants of integration, $\lambda_{01}(F) = -\lambda_{11}(-F) = 2 + F - f$, and $\{\lambda_{0k}(F) = -\lambda_{1k}(-F), k = 2,3,4\}$ is the set of roots of the algebraic equation

$$\lambda(F)[\lambda(F) - (2 + F + f)][\lambda(F) - (2 + F)] - \nu D\left[\lambda(F) - \left(2 + F + \frac{f}{2}\right)\right] = 0. \tag{13}$$

Ten independent conditions for the ten constants of integration $C_{ik}, k = 0,1,2,3,4$, and for the probability current J can be determined at the points of discontinuity, by requiring continuity and periodicity for the quantities $\mathbf{P}_i^s(x)$ and $j_{ni}(x)$, i.e.,

$$P_{n0}^{s}\left(\frac{1}{2}\right) = P_{n1}^{s}\left(\frac{1}{2}\right), \quad P_{n0}^{s}(0) = P_{n1}^{s}(1),$$

$$j_{n0}\left(\frac{1}{2}\right) = j_{n1}\left(\frac{1}{2}\right), \quad j_{n0}(0) = j_{n1}(1), \quad n = 1,2,3. \tag{14}$$

As it follows from Eq. (7) that $J = \text{const}$ [see also Eq. (8)], the system of linear algrebaic equations (14) contains only ten linearly independent equations. By including an eleventh (normalization) condition,

$$\sum_{n=1}^{3} \int_0^1 P_n^s(x)dx = 1, \tag{15}$$

a complete set of conditions is obtained for the ten constants of integration and for the probability current J. This procedure leads to an inhomogeneous set of eleven linear algebraic equations. Now, J can be expressed as a quotient of two determinants of the eleventh degree. The exact formula, being complex and cumbersome, will be presented in the Appendix [Eq. (A1)]. It will be used to find (i) the dependence of the current J on the tilting force F and the dependence of the mobility $m = dJ/dF$ on the switching rate ν, which are displayed in Fig. 2(a) and Fig. 4, respectively, and (ii) the asymptotic limits of the current at a low temperature, fast-noise, and adiabatic noise. The behavior of J at different system parameters regimes will be considered in Sec. III. All numerical calculations are performed by using the software MATHEMATICA 4.1.

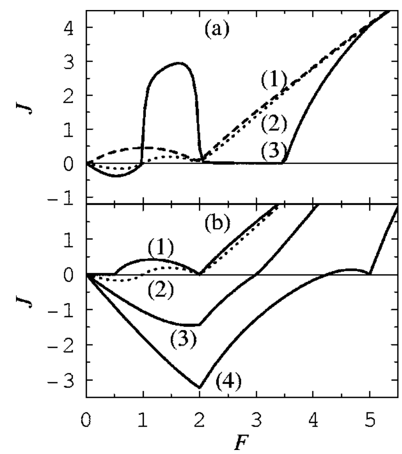

FIG. 2. The current J vs the applied force F in the region of anomalous mobility. (a) The general case [Eq. (A1)]. Dashed line (1): $D = 10^{-2}$, $\nu = 10^{-2}$, and $f = 1.9$. Dotted line (2): $D = 10^{-2}$, $\nu = 10^{-2}$, and $f = 3$. Solid line (3): $D = 10^{-7}$, $\nu = 10^{3}$, and $f = 3$. Note that in the cases of curves (2) and (3) the phenomenon of absolute negative mobility occurs. (b) The case of an adiabatic limit [Eqs. (A3)–(A7)]. Here the temperature D is zero and the curves correspond to (1) $f = 1.5$, (2) $f = 3$, (3) $f = 5$, and (4) $f = 7$. All quantities are dimensionless with scaling by Eqs. (4) and (5).

III. RESULTS

A. Anomalous resistance

The quantities of central interest are the average particle current J and its response to an applied force F, i.e., the differential mobility at $F = F_0$,

$$m|_{F=F_0} = \left.\frac{dJ(F)}{dF}\right|_{F=F_0}. \tag{16}$$

In the case of a small applied force,

$$m_0 = \left.\frac{dJ(F)}{dF}\right|_{F=0} \tag{17}$$

will be called mobility.

For the general case, the exact equation (A1) for the current J enables us to establish a number of effects characterizing anomalous behavior of the resistance (or mobility).

Later Fig. 2(a) illustrates the behavior of the current J as a function of the tilting force F in the region of anomalous resistance. It appears that the curves are highly nonlinear. For curves (2) and (3), the phenomenon of absolute negative mobility occurs: the particle moves in the direction opposite to a small external force F. Moreover, all curves exhibit intervals of F where particle speed decreases as the applied drive is increased—an effect that is termed negative differential resistance. In the case of curve (3) two additional effects occur. First, one can see a hypersensitive response—the jumps of the current at $F = 1$ and $F \approx 2$. Second, at a low temperature and a large switching rate ν the current exhibits characteristic "disjunct windows" of the tilting force [$2 < F < 3,5$ for curve (3)], where the value of the current is very small.

B. The adiabatic limit in the case of low temperatures

In this section we examine a simple, yet physically important case of the adiabatic limit at low temperatures that captures some features of the general system, including absolute negative mobility and negative differential resistance. We start from the asymptotic formulas (A3)–(A7) for the current J reproduced in the Appendix. In Fig. 2(b) we plot a series of curves of J vs the tilting force F at various noise amplitudes f, showing that four different types of the graphs $J(F)$ emerge. These four types of $J(F)$ correspond to four characteristic regions for the parameter f: $f<2, 2<f<4, 4<f<6$, and $f>6$. (i) If $f<2$, the current is zero for $F\leqslant 2-f$ and at the point $F=2$; for $2-f<F<2$ the positive current exhibits a bell-shaped extremum, and if $F>2$, then J increases monotonically from zero. (ii) In the case of $2<f<4$, the current exhibits a reversal of the direction at $F=f-2$, if $F=2$, then $J=0$; the current reaches two local minima and a local maximum at a finite F. (iii) For $4<f<6$, the behavior of J is characterized by current reversals at $F=f-2$ and by one local extremum (minimum); no negative differential resistance for a positive current occurs. (iv) If $f>6$, the current behaves similarly as in the case of (ii); however, the zeros of J appear at the values of the tilting force $F=[2(2+f)]^{1/2}$ and $F=f-2$. Note that negative differential resistance for a positive current disappears only in the case of $4<f<6$, while absolute negative mobility is absent only in the case of $f<2$. The existence of four characteristic regions for f and the corresponding behavior of $J(F)$ are closely related to configurations of the net potentials $V_n(x)=V(x)-Fx-z_nfx$ associated with the right-hand side of Eq. (6). For example, in case $f<2$ and $0<F<2-f$, the net potentials $V_n(x)$ for all states $n=1,2,3$ of the nonequilibrium noise Z have minima at $x=1/2$. Hence, at a zero temperature in a stationary state the particles are locked in the potential minima and there is no current.

Let us note that the sufficient and necessary condition $f>2$ for the existence of absolute negative mobility has a distinct physical meaning: for $f>2$ and $F<f-2$, the net potentials $V_n(x)$ in the states $n=1,3$ are monotonous functions [also see Fig. 1(a)] and hence in the state $n=1$ the particles will move in the negative direction. In this case, absolute negative mobility is possible and can be physically understood. Let us take a closer look at the latter statement. The adiabatic limit occurs when $\nu\to 0$. In the low frequency domain there is, in each of the three states, enough time for the probability distribution to relax to and spend most of the time in the stationary probability distribution belonging to that state. For a sufficiently low temperature $D\ll 1$ the probability distribution in the $n=2$ state can be approximated by the Boltzmann distribution,

$$P_2^s(x) = C_2 \exp\left[-\frac{V_2(x)}{D}\right], \qquad (18)$$

where C_2 is the appropriate normalization constant. In the state $n=2$ the current J_2 is zero. The current J_3 in the state $n=3$ and the current J_1 in the state $n=1$ are

$$J_1 = 2\int_0^{1/2} P_1^s(x)(2-f+F)dx = -2\int_{1/2}^{1} P_1^s(x)(f+2-F)dx, \qquad (19)$$

$$J_3 = 2\int_0^{1/2} P_3^s(x)(2+f+F)dx = 2\int_{1/2}^{1} P_3^s(x)(f+F-2)dx. \qquad (20)$$

Equations (19) and (20) are consistent with the fact that the flux of particles is continuous.

For the total current we have

$$\begin{aligned} J &= 2[p_{11}(2-f+F) + p_{31}(2+f+F)] \\ &= 2[p_{32}(f+F-2) - p_{12}(f+2-F)], \end{aligned} \qquad (21)$$

where $p_{n1}\equiv\int_0^{1/2} P_n^s(x)dx$, $p_{n2}\equiv\int_{1/2}^{1} P_n^s(x)dx$.

From Eq. (18) we obtain

$$p_{22} = p_{21}\frac{F+2}{2-F}, \quad F<2, \qquad (22)$$

i.e., upon the application of a tilting force $F>0$ the particles will preferentially reside on the right-hand side of the net potential $V_2(x)$, facilitating entry into the state $n=1$ where the motion is biased into the negative direction. Now, using the stationarity conditions $p_{21}=p_{31}$, $p_{12}=p_{22}$ [see Eq. (9)], and the normalization condition (15) we get the formula

$$J = -\frac{fF[(f-2)^2 - F^2]}{(f-2)(3f-2) - F^2} < 0, \qquad (23)$$

with $f>2$ and $F<\min\{f-2,2\}$. Thus, we have obtained a result of the Appendix, namely, Eq. (A6). The effect of ANM appears because of a regulative role of the state $n=2$. The crucial observation now is that the application of a small tilting force F along the x axis will favor the concentration of particles on the right-hand side of the net potential $V_2(x)$ that is characterized by a flatter slope than the left-hand side. The negative current happens because of the transitions between the states $n=2$ and $n=1$; hence, negative mobility can be expected if this effect is larger than the linear effect of a direct response to the tilting force. Note that an analogous procedure can be repeated in a straightforward way for all six different net potential $V_n(x)$ configurations that appear by the variation of the parameters f and F. The corresponding results coincide with the formulas (A3)–(A7).

Next we shall consider the phenomenon of negative differential resistance (NDR) for a positive current. In the case of $2<f<4$ and $f-2<F<2$ the corresponding net potential $V_n(x)$ configuration is shown in Fig. 1(b). At a low temperature, $D\to 0$, a particle cannot move freely along the potentials V_1 and V_2. However, if switching is allowed between the potentials V_n, $n=1,2,3$, the particle will move downhill along the potential V_3 and the total current is positive. The stationary probability densities $P_1^s(x)$ and $P_2^s(x)$ for the states $n=1,2$ are determined by the Boltzmann distributions [also see Eq. (18)]. The physical mechanism for NDR is evident from the considerations mentioned above at Eq. (23). As the tilting force F increases, the slope of the right-hand side of the potential $V_2(x)$ decreases and the fraction of particles locked in the state $n=1$ and on the right-hand side of the potential V_2 increases. Consequently, the fraction of the particles in the state $n=3$, $(p_{31}+p_{32})\sim(2-F)$, decreases and the total current $J(F)=2p_{31}(2+f+F)$ versus F is a decreasing function, at least in the case when F is great enough ($F \approx 2$). The corresponding formula for J is represented in the Appendix [Eq. (A3)]. Now, from Eq. (A3) the differential resistance at $F=2$ can be derived:

$$m|_{F=2} = -\frac{(16-f^2)}{2(8-f)} < 0, \quad 2 < f < 4.$$

If $f>2$, the phenomenon of NDR different from those considered above can be observed in the tilting force interval,

$$0 < F < \{(f-2)[2(2f-1) - [f(13f-8)]^{1/2}]\}^{1/2}.$$

In this case NDR is associated with ANM and the current is negative [see Fig. 2(b)].

C. Absolute negative mobility

Next we will discuss the effect of absolute negative mobility. At a long-correlation-time limit $\nu\to 0$, Eqs. (7) for $P_1^s(x)$, $P_2^s(x)$, and $P_3^s(x)$ are decoupled and the total current is given by the average of each current for the corresponding potential configurations. The result is represented in the Appendix [Eq. (A2)]. In this case, our analytical solution for the mobility (17) of the system (6)reads as

$$m_0 = \frac{(2-f)\{2\beta - 3 - f + \alpha[f + 4(2+f)\gamma/(1-2\beta)]\}}{\alpha[3f - 2 + 2(6-f)\beta]}, \tag{24}$$

where

$$\alpha = \left[2D \sinh\left(\frac{1}{2D}\right)\right]^2,$$

$$\beta = \frac{4D(e^{(2-f/2D)} - 1)(1 - e^{-(2+f/2D)})}{(f^2-4)(1-e^{-f/D})},$$

$$\gamma = \frac{2}{f^2-4} + \beta\left[\frac{2f}{f^2-4} + \frac{1+e^{-f/D}}{2D(1-e^{-f/D})}\right].$$

In the high temperature limit $D \gg 1$, the mobility tends to 1 and the system does not exhibit ANM. In the case of zero temperature, $D=0$, one finds from Eq. (24) that in the assumption $f>2$ the mobility equals

$$m_0 = \frac{-f(f-2)}{3f-2}, \tag{25}$$

supporting ANM in agreement with Figs. 2 and 3. If $f<2$, we can see that m_0 tends to zero as $D\to 0$. This result is consistent with the physical intuition that at deterministic stationary states (minima of potentials) the probability densities $P_n^s(x)$ are δ distributed: the random variable Z takes values $z_n=1,0,-1$ for a sufficiently long time to allow the deterministic stationary state to be formed. Figure 3 shows the mobility m_0 as a function of the temperature D for four different values of the noise amplitude f in the case of a low switching rate limit. In this figure, one can also observe absolute negative mobility at small and intermediate temperatures, which apparently gets more and more pronounced as the noise amplitude $f>2$ increases. The tendency apparent in Fig. 3, namely, an increase of the critical temperature D_c (the temperature at which the phenomenon of ANM disappears) as the amplitude f grows, also takes place in the case of large f, e.g., if $f>6$. At sufficiently large values of f, $f\gg 2$, the formulas for the leading-order terms of the mobility m_0 and of the critical temperature are

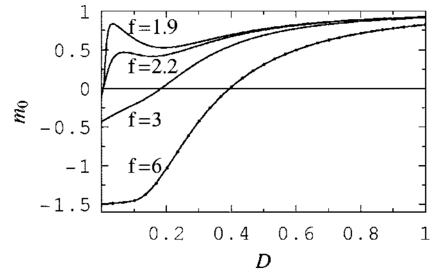

FIG. 3. The mobility m_0 vs temperature D at various noise amplitudes f in the case of an adiabatic limit [Eq. (24)]. Note that in the case of $f<2$ the mobility is positive for all values of $D>0$. At large values of the temperature, $D>1$,the mobility m_0 saturates to the value 1. The dots were computed by means of the exact equation (A1) with $f=6$, $\nu=10^{-6}$. All quantities are dimensionless with scaling by Eqs. (4) and (5).

$$m_0 \approx \frac{1}{\alpha}\left[1 - \frac{f}{3}(\alpha-1)\right], \quad D \ll f; \quad D_c \approx \frac{1}{6}\sqrt{f}. \tag{26}$$

In Fig. 3 we can see that the asymptotic formula (26) for D_c is in agreement with the exact result for $f=6$. It is remarkable that the phenomenon of ANM occurs also at high temperatures, if only the noise amplitude f is large enough. The physical mechanism appropriate to generate results (26) is analogous to those considered in Sec. III B. The key-factor is the stationary probability distribution $P_2^s(x)$ in the state $n=2$. However, as the temperature is relatively high, causing significant diffusion over the potential barrier, the stationary probability density $P_2^s(x)$ is not determined by the Boltzmann distribution and in the state $n=2$ the fractional current $J_2 \neq 0$. Instead of Eq. (18), the stationary probability density is given as

$$P_2^s(x) = Ce^{-V_2(x)/D}\int_x^{1+x} e^{V_2(y)/D}dy, \tag{27}$$

where C is the appropriate normalization constant.

Focusing on the small tilting force, $F\to 0$, we obtain that $p_{21}\approx p_{22}\{1-F[1-(1/\alpha)]\}$ and $J\approx(1/3\alpha)F+2[p_{32}(f+F-2) - p_{12}(f+2-F)]$, which, by virtue of Eqs. (9), (15), (19), and (20), yields the expression (26). The destructive influence of temperature for ANM is twofold. First, the thermal diffusion generates a positive current $J_2 \approx (1/3\alpha)F$ in the state $n=2$ and, second, in agreement with physical intuition, the asymmetry of particle distribution $1-(p_{21}/p_{22})=F[1-(1/\alpha)]$ decreases as temperature D grows.

Figure 4 shows a plot of m_0 as a function of the switching rate ν at various temperatures. For low temperatures, $D\to 0$, and $f>2$, the mobility is approximately given by

$$m_0 = -\frac{\nu(f-2)(f+4)\{16\tilde{\alpha}(2+f)^3 + \nu(\tilde{\alpha}+1)(4+f)[f^2+4f+8]\}}{8\tilde{\alpha}(f+2)^2[3\nu f(f+4) - 32\tilde{\alpha}(f-2)]}, \tag{28}$$

with

$$\tilde{\alpha} = \exp\left[-\frac{\nu(f+4)}{8(f+2)}\right] - 1,$$

[also see Eq. (A9)]. In this case the mobility is always negative and m_0 decreases monotonically from $m_0=-f(f-2)/(3f-2)$ to the value

$$m_0 = -\frac{2(f^2-4)}{3f}, \tag{29}$$

as the switching rate grows.

It is remarkable that the absolute value of m_0 increases as $\nu\to\infty$, $\nu D \ll 1$. This phenomenon can be physically understood, taking into account that the fraction of particles effectively locked in the potential minimum of the state $n=2$ reads as

$$\rho = -\frac{8\tilde{\alpha}(f-2)(f+4)}{3\nu f(4+f) - 32\tilde{\alpha}(f-2)}. \tag{30}$$

More precisely, for $n=2$, the parameter ρ is the probability that the position of particles coincides with the deterministic stationary state $x=1/2$ (a stationary stable point in the absence of noise). At the adiabatic limit $\nu\to 0$ the parameter $\rho=(f-2)/(3f-2)$. For $\nu\to\infty$ we get $\rho\approx 8(f-2)/3f\nu$; thus the locked fraction of particles decays algebraically to zero in ν^{-1}.

Two important asymptotic regimes occur in the $D\neq 0$ situation: first, the regime of low diffusion levels $D\nu \ll 1$ for which the characteristic distances of thermal diffusion $\sqrt{D\tau_c}$ are much smaller as the typical deterministic distances of the driven particles during the noise correlation time $\tau_c=1/\nu$, and, second, the regime $D\nu \gg 1$, dominated by thermal diffusion. In the regime of low diffusion the temperature is small enough, $D<D_c\approx\sqrt{f}/6$, for the phenomenon of ANM to appear, contrary to the case of $D\nu\gg 1$, where the mobility is positive. At the fast-noise limit, $\nu\to\infty$, $D\nu\to\infty$, the mobility m_0 can be easily found from the asymptotic formula (A8) represented in the Appendix. Finally, a most salient intermediate regime occurs in which the mobility exhibits a resonant-like enhancement at finite values of ν (also see Fig. 4). For example, in Fig. 4 the curve $m_0(\nu)$ with $D=0.01$ shows an amplification of ANM at $\nu=18$.

D. Hypersensitive differential response

Next we will discuss both the effect of "disjunct windows" (DW) and the effect of hypersensitive differential response (HDR). Both effects appear at large values of the switching rate ν and low values of the temperature D. Figure 5 exhibits the behavior of the current J versus the tilting force F in the region of an anomalous differential response. Here the curve (5) demonstrates the DW effect. As already mentioned, at large values of ν the current exhibits a characteristic disjunct zone of the tilting force, $2+(f/2)>F>\max\{2,f-2\}$, in which the current is very small. The necessary conditions for the existence of the DW effect are the regime of a small diffusion $D\nu\ll 1$ and a large ν. Hence at a low temperature, $D\to 0$, the DW effect can analytically be examined by Eq. (A10). In this case for $2+(f/2)>F>\max\{2,f-2\}$ the current is exponentially small, $J\sim \nu e^{-\nu d}$ with $d-a$ constant.

As the diffusion is negligible the physical mechanism for DW is simple. For the described interval of F the net potential $V_1(x)$ exhibits a minimum and the potentials $V_2(x)$, $V_3(x)$ are monotonic functions. If the correlation time of the noise $\tau_c=1/\nu$ is small enough, a particle in the state $n=2$ cannot, before switching to the state $n=1$, move to the next spatial period and, consequently, in the stationary state all particles are concentrated at the potential well $V_1(x)$ and on the right-hand side of the potential $V_2(x)$. This is because the absolute value of the deterministic velocity of particles on the right-hand side of $V_1(x)$ is greater than the velocity on the right-hand side of $V_2(x)$. It is obvious that the total current J tends

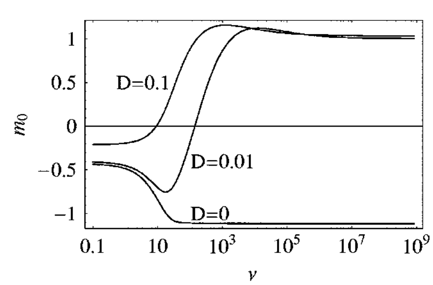

FIG. 4. The mobility m_0 vs the switching rate ν at various temperatures D and $f=3$. The curves with $D=0.1$ and with $D=0.01$ are computed from the exact formula (A1) for the current J. The line with $D=0$ corresponds to the asymptotic equation (28). All quantities are dimensionless with scaling by Eqs. (4) and (5).

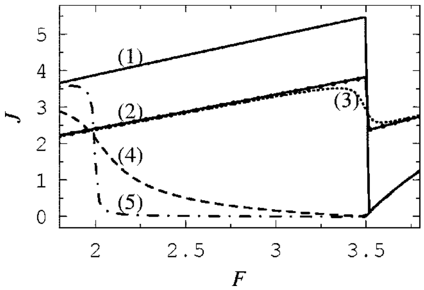

FIG. 5. The current J vs applied force F in the region of negative differential resistance. The curves are computed from the exact equation (A1) at $f=3$. Solid line (1): $D=3\times10^{-9}$, $\nu=10^{8}$. Solid line with filled dots (2): $D=10^{-6}$, $\nu=10^{12}$. Dotted line (3): $D=5\times10^{-3}$, $\nu=10^{9}$. Dashed line (4): $D=10^{-5}$, $\nu=10^{3}$. Dashed-dotted line (5): $D=3\times10^{-9}$, $\nu=10^{4}$. The filled dots were computed by means of the asymptotic formula (A8). Note the hypersensitive response (jumps) of the current in the cases of curves (1), (2), and (5). All quantities are dimensionless with scaling by Eqs. (4) and (5).

to zero as $\nu\to\infty$, since the trapping probability in $V_1(x)$ and $V_2(x)$ tends to 1.

The described scheme is valid only for low temperatures. Otherwise, a particle is able to pass potential barriers in both directions by a thermally activated escape. However, it predominantly moves to the right and a positive current occurs.

In Fig. 5, the curve (5) demonstrates also the phenomenon of HDR, i.e. jumps of the current with a large derivative for some values of the tilting force F, at $F=2$. The jump of J at $F=2$ can estimated with the help of the following physical considerations. As $\nu\to\infty$, $\nu D\ll 1$ and $D\to 0$, the diffusion is very small and the actual three-level net potential configuration is equivalent to a two-level average net potential configuration V_{avg1} and V_{avg2}. The potential V_{avg1} is characterized by the effective forces $2+(f/2)+F$ and $f+F-2$ in the left-hand and right-hand sawtooth sides, respectively. The net potential V_{avg2} has a local minimum at $x=1/2$ with the corresponding effective forces $F+2-f$ and $F-2-(f/2)$. Due to very small diffusion the fraction of particles in the potential well V_{avg2} is negligible and this leads to the circumstance that the formation of the current is determined by V_{avg1}. So the current J can be found, by using the continuity of particle flux and the normalization condition, from the equations

$$J = 2p_2(f+F-2), \quad p_1\left(2+F+\frac{f}{2}\right) = p_2(f+F-2),$$

$$p_1 + p_2 = 1, \tag{31}$$

where p_1 and p_2 are the stationary probabilities that a particle is located in the intervals $x\in(0,1/2)$, and $x\in(1/2,1)$, respectively. Fom Eqs. (31) we obtain that $J|_{F=2-\varepsilon}=2f(f+8)/(3f+8)+O(\varepsilon)$ and one finds for the jump of the current:

$$\Delta J|_{F=2} = J|_{F=2+\varepsilon} - J|_{F=2-\varepsilon} = -\frac{2f(f+8)}{3f+8}, \tag{32}$$

with $\varepsilon\to 0$. Note that this result can be obtained also straightforwardly from Eq. (A11). Moreover, from Eq. (A11) it can be easily shown that

$$m|_{F=2} \approx -\frac{\nu(4-f)(8+f)}{32(8-f)}, \tag{33}$$

with $2<f<4$, $D\to 0$. An extreme sensitivity of the differential negative mobility $m|_{F=2}$ to the switching rate ν can be seen from the factor ν in Eq. (33)—the absolute value of the mobility $m|_{F=2}$ increases unboundedly as $\nu\to\infty$. Furthermore, in the case considered here ($2<f<4$, $D\nu\ll 1$, $D\to 0$) the effect of HDR also occurs at $F=f-2$ [Fig. 2(a)]. According to Eq. (A11) for $F\to f-2$, the formula for the leading-order term of the differential mobility reads as

$$m|_{F=f-2} = \frac{3\nu(4-f)}{8f}.$$

At the fast-noise limit, $\nu\to\infty$, the corresponding jump of the current can be estimated by the equation

$$\Delta J|_{F=f-2} = \frac{12f(f-2)}{7f-8}.$$

To illustrate the phenomenon of HDR in the "strong diffusion" domain, $D\nu\gg 1$, we are restricted to the low temperature case, $D\ll 1$. At the fast-noise limit, $\nu\to\infty$, one finds from Eq. (A8) that the differential mobility $m|_{F=2+(f/2)}$ behaves asymptotically as

$$m|_{F=2+f/2} \approx -\frac{f(f+8)}{2D(3f+8)^2}, \quad 0<f<8. \tag{34}$$

It is obvious that the absolute value of the differential mobility $m|_{F=2+(f/2)}$ increases unboundedly as $D\to 0$. The corresponding jump of the current is $\Delta J|_{F=2+(f/2)}=-4f(8+f)/(f+4)(3f+4)$. The dependence of the current $J(F)$ on the tilting force F for a fixed switching rate $\nu=10^{12}$, a fixed temperature $D=10^{-6}$, and $f=3$ is shown in Fig. 5 as curve (2). We can see that the asymptotic formula (A8) is in excellent agreement with the exact results. According to numerical calculations from the exact equations (12)–(15) we emphasize that the effect of HDR at $F=2+(f/2)$ is extremely pronounced in an intermediate regime, $(1/D)\sim\nu\to\infty$, with the current picking up from zero [also see curve (1) in Fig. 5]. It is remarkable that in the case of fast noise the value of the tilting force $F=2+(f/2)$ corresponds to the critical value of F at which the average net potentials configuration changes.

The appearance of HDR is not confined to the cases described above. If $\nu\to\infty$ and $D\to 0$, the phenomenon of HDR can occur, depending on particular values of the parameters νD and f, at $F=2+(f/2)$, 2, $f-2$. For example, in the case of $2<f<8$, $D\nu\gg 1$, we have $m|_{F=f-2}\approx f(f-2)/6D(2f-1)^2$. Note that all these values of F correspond to a change in the net potentials configuration.

IV. CONCLUDING REMARKS

Above, we have presented some exact and asymptotical results for the dynamics of an overdamped Brownian particle in a periodic, symmetric, one-dimensional sawtooth potential landscape subjected to a static tilting force and to both thermal noise and spatially nonhomogeneous three-level colored noise. A major virtue of the proposed model is that an interplay of three-level colored and thermal noises in tilted ratchets with simple symmetric sawtooth potentials can generate a rich variety of cooperation effects, namely, absolute negative mobility (ANM), negative differential resistance (NDR), hypersensitive differential response (HDR), and the phenomenon of "disjunct windows" (DW) for the tilting force.

For both slow and fast fluctuating forces, and for low temperatures, we have presented analytical approximations that agree with the exact numerical results. One of our major results is a resonant-like enhancement of absolute negative mobility at intermediate values of the switching rate of nonequilibrium noise (also see Fig. 4). Two circumstances should be pointed out. (i) A resonant-like behavior can occur in a system parameters domain where the characteristic distance of thermal diffusion $\sqrt{D/\nu}$ is comparable with typical deterministic distances for the driven particles during the noise correlation time. (ii) There is an upper limit temperature D_c beyond which the phenomenon of ANM disappears. Notably, at increasing the noise amplitude f the critical temperature D_c grows as $D_c \sim \sqrt{f}$, [see Eq. (26)]. It is obvious that the presence and intensity of ANM can be controlled by a thermal noise (also see Fig. 3). The advantage of this model is that the control parameter is temperature, which can easily be varied in experiments. Moreover, as in Eq. (5) the friction coefficient κ is absorbed into the time scale, so, in the original (unscaled) set-up, the particles of different friction coefficients are controlled by different switching rates. According to the suggestions in [4,7,8,29], this can lead to an efficient mechanism for the separation of different types of particles by exploiting the sensitive dependence of the current-load characteristics on the switching rate (also see Fig. 4 and Ref. [18]).

The phenomenon of ANM in systems similar to ours have been studied in [19]. However, in contrast to ours, in those models the authors choose a symmetric potential $V(x)$ with two minima per period. Perhaps the most fundamental difference is that in the models of [19] unbiased transitions can take place between the discrete states only at the minima of potentials. As a consequence the dependence of the current on the switching rate disappears.

Our another major result is establishing the effects of both HDR and DW at large values of the switching rate ν and low values of the temperature D [also see Fig. 5 and Eqs. (33) and (34)]. We emphasize that our mechanism of HDR is of a qualitatively different nature from a recently found effect, where a noise-induced enhancement of the current of Brownian particles in a tilted ratchet system has also been established [13,14]. In the mechanism reported here hypersensitivity is achieved by a combined influence of fast nonequilibrium noise and a tilt-force-induced change of the net potentials configuration. It should be pointed out that in the present model the effect of HDR is pronounced in the case of a fast switching of thenonequilibrium noise, while in the models of Refs. [13,14] the hypersensitive transport is generated by low or moderate values of the switching rate.

It is quite remarkable that the results of HDR seem to be applicable for amplifying adiabatic time-dependent signals $\omega(t)$, i.e., signals of much longer periods than the characteristic time of establishing a stationary distribution, even in the case of a small input signal-to-noise ratio $|\omega(t)|/\sqrt{D} \ll 1$. For example, in the case described by the formula (34), i.e., the tilting force $F=2+\frac{1}{2}f+\omega(t)$, the system may be able to amplify an ultrasmall deterministic ac signal $\omega(t)$ up to the value of the order of unity (cf. [12,13]). This conjecture presents an objective that is worthwhile to be addressed in greater detail in the future.

Surprisingly enough, at a low temperature and a large switching rate, $D\nu \ll 1$, the current is very small in the finite interval of the tilting force, $2+\frac{1}{2}f > F > \max\{2, f-2\}$. This novel feature for a Brownian particle is, so far, mainly of theoretical interest while applications are not clearly identifiable yet.

Finally, we believe that the model discussed here is particularly suitable for an experimental realization along the lines described in Ref. [19], e.g., for particles suspended in a hydrodynamic flow.

ACKNOWLEDGMENTS

This work was partly supported by the Estonian Science Foundation Grant No. 5943 and the International Atomic Energy Grant No. 12062, for which the authors extend their gratitude.

APPENDIX: FORMULAS FOR THE CURRENT

Here the exact formula for the current J and some asymptotic formulas following from Eqs. (11)–(15) will be represented.

1. The general case

From Eqs. (11)–(15) one can conclude that the current J is given by

$$J = \frac{\det[B_{lr}(1-\delta_{r,2}) + \delta_{l,11}\delta_{r,2}]}{\det(B_{lr})}, \tag{A1}$$

where the matrix $(B_{lr}), l, r = 1, \ldots, 11$ is defined as follows:

$$B_{n+3\,2} = B_{n2} = \frac{\delta_{n,1}}{F+2-f} - \frac{\delta_{n,3}}{F+f-2},$$

$$B_{n+3\,2i+1} = B_{n2i+1} = 1 - 3\delta_{n,2i+1}\frac{h_i+F}{h_i+F+z_n f},$$

$$B_{n+6\,2} = \delta_{n,1} - \delta_{n,3},$$

$$B_{n+6\,2i+1} = (F+h_i+z_n f)B_{n2i+1},$$

$$B_{10\,2} = 1, B_{10\,2i+1} = (h_i+F-f)B_{1\,2i+1},$$

$$B_{11\,2} = \frac{F}{F^2-(f-2)^2}, B_{11\,1} = -\frac{3f}{2(2+F-f)},$$

$$B_{11\,3} = -\frac{3f}{2(F-2+f)},$$

$$B_{n\ k+3+4i} = A_{nik} \exp\left[\frac{\lambda_{ik}(F)}{2D}\right],$$

$$B_{n+3\ k+3+4i} = A_{nik} \exp\left[\frac{\lambda_{ik}(F)}{D}\delta_{i,1}\right],$$

$$B_{n+6\ k+3+4i} = [h_i + F + z_n f - \lambda_{ik}(F)]B_{n\ k+3+4i},$$

$$B_{10\ k+3+4i} = [h_i + F - f - \lambda_{ik}(F)]B_{1\ k+3+4i},$$

$$B_{11\ k+3+4i} = \frac{D}{\lambda_{ik}(F)}\sum_{n=1}^{3}(B_{n\ k+3+4i} - B_{n+3\ k+3+4i}),$$

with $n=1,2,3$; $i=0,1$; $k=1,2,3,4$; $h_0=2$, $h_1=-2$, $z_1=-1$, $z_2=0$, $z_3=1$, and the quantities A_{nik}, $\lambda_{ik}(F)$ are the same as in Eq. (12).

2. The adiabatic limit

At the adiabatic limit $\nu \to 0$ the form of the leading term of the stationary current is

$$J = \frac{2[F^2 - 4 + B(F) + B(-F)]}{4[F - 4A(0,F)] + C(F) - C(-F)}, \qquad \text{(A2)}$$

where

$$B(F) = (F+2)(F-2-f)\frac{1-2A(0,F)}{1-2A(f,F)},$$

$$C(F) = B(F)\frac{1+2A(f,F)}{2+F-f},$$

$$A(f,F) = \frac{4D\{\cosh(1/D) - \cosh[(f-F)/2D]\}}{[(f-F)^2-4]\sinh[(f-F)/2D]}.$$

In the case of low temperature, $D \to 0$, the Equation (A2) reduces to more simple formulas. Three characteristic regions can be discerned for the parameter f.

(i) If $f<2$, then there is no current as $F<2-f$, $J=0$. For $2>F>2-f$ we have

$$J = \frac{(2+f+F)(2-F)[F^2-(f-2)^2]}{16f-(4-F)[f^2+4-F^2]}. \qquad \text{(A3)}$$

If $2<F<f+2$, then

$$J = \frac{(f+2F+4)(F-2)[F^2-(f-2)^2]}{2\{F[F^2-(f-2)^2]+F^2+f^2-4\}}. \qquad \text{(A4)}$$

For $F>f+2$ the following formula is valid:

$$J = \frac{(3F^2-4f-12)[F^2-(f-2)^2]}{F[8f+3(F^2-f^2-4)]}. \qquad \text{(A5)}$$

(ii) In the case of $2<f<4$ and $F<f-2$, the current J can be given as

$$J = -\frac{fF[(f-2)^2-F^2]}{(f-2)(3f-2)-F^2}. \qquad \text{(A6)}$$

If $F>f-2$, then the behavior of J is determined by Eqs. (A3)–(A5).

(iii) For $f>4$ and $2<F<f-2$, one finds from Eq. (A2) that

$$J = \frac{2[F^2-(f-2)^2][F^2-2(f+2)]}{F[3(F^2-f^2-4)+8f]}. \qquad \text{(A7)}$$

In the intervals $0<F<2$ and $f-2<F<\infty$ the form of J is given by Eq. (A6) and by Eqs. (A4) and (A5), respectively.

3. The fast-noise limit

In the fast-noise limit, we allow ν to become large, $\nu \to \infty$, holding all other parameters fixed. Thus, if $\nu D \to \infty$, in the large ν limit the current can be given as

$$J = \frac{[F^2-(f-2)^2][(4+f)^2-4F^2][\tilde{A}(F)-\tilde{A}(-F)]}{24D[8F^2-(f-2)(f^2-16)]+[\tilde{B}(F)+\tilde{B}(-F)][F^2-(f-2)^2][(4+f)^2-4F^2]}, \qquad \text{(A8)}$$

where

$$\tilde{A}(F) = (2-F)\{(f-2-F)\alpha(F) - [f+2(2+F)]\beta(F)\},$$

with

$$\alpha(F) = \left[\exp\left(\frac{2-f+F}{2D}\right)-1\right]^{-1}, \beta(F) = \left[\exp\left(\frac{2(2+F)+f}{4D}\right)-1\right]^{-1},$$

and

$$\tilde{B}(F) = (f+2F)\beta(F) - (f-F)\alpha(F).$$

4. The zero-temperature case

In the asymptotic limit of low temperature, $D \to 0$, we find that for $F < \min\{2, f-2\}$, $f > 2$, the current behaves asymptotically as

$$J = \frac{2\nu[(4+f)^2 - 4F^2][\tilde{\alpha}(F)(4+f+2F)(f-2+F) - \tilde{\alpha}(-F)(4+f-2F)(f-2-F)]}{\nu[(4+f)^2 - 4F^2][\tilde{\alpha}(F)(3f+4F) + \tilde{\alpha}(-F)(3f-4F)] - 16\tilde{\alpha}(F)\tilde{\alpha}(-F)(f+4)}, \tag{A9}$$

where

$$\tilde{\alpha}(F) = (2-F)^2(f-2-F)\left[\exp\left(\frac{-\nu(4+f-2F)}{4(2-F)(2+f-F)}\right) - 1\right].$$

For $f+2 > F > \max\{2, f-2\}$ and $f > 2$, the formula for the leading-order term of the current is

$$J = \frac{2\nu[(4+f)^2 - 4F^2][4+f-2F][F^2 - (f-2)^2]}{\nu(2+F-f)(4+f-2F)[8F(2-f-F) + 3f(2+f+F)] + 4(2+F)(8-f)\tilde{\beta}(F)}, \tag{A10}$$

with

$$\tilde{\beta}(F) = \frac{(2+f-F)^2(f-2+F)}{(2-F)^2(f-2-F)}\tilde{\alpha}(F).$$

If $2 < f < 4$ and $f-2 < F < 2$, then

$$J = \frac{2\nu(2-F)(4+f+2F)^2[F^2 - (f-2)^2]}{\nu(2-F)(4F+3f)(F+2-f)(4+f+2F) - 4\tilde{\alpha}(-F)[F(f+16) - f^2 - 2f + 32]}. \tag{A11}$$

We note that formulas (A9)–(A11) can also be applied in the case of the fast-noise limit, $\nu \to \infty$, by the assumption that $\nu D \ll \min\{(F-2)^2, (2+f-F)^2\}$.

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